Note Packet #4

Total Probability & Bayes Theorem

CE 3710

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Partition of Sample Space

\{E_1, E_2, \ldots, E_n\} is a partition of the sample space \(S\)
if the events are *mutually exclusive & collectively exhaustive.*

i.e. every point in \(S\) is contained in one and only one \(E_i\)
Law of Total Probability

$$P(A) = \sum_{i=1}^{n} P(A \mid E_i)P(E_i)$$

**Proof:**

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup ... \cup (A \cap E_n)$$

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + ... + P(A \cap E_n)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$= \sum_{i=1}^{n} P(A \mid E_i)P(E_i)$$
Example:

Suppose I have two coins.

  One is FAIR.

  The other has TWO heads.

If I randomly select a coin and flip it TWICE,
What is probability of getting heads both times?
Bayes Theorem

If \( \{E_1, E_2, \ldots, E_n\} \) is a partition of the sample space \( S \) with \( P( E_i ) > 0 \) for \( i = 1, \ldots, n \)

Then for any event \( A \) for which \( P(A) > 0 \):

\[
P(E_k | A) = \frac{P(A | E_k)P(E_k)}{\sum_{i=1}^{n} P(A | E_i)P(E_i)}
\]
Bayes Theorem

\[ P(E_k | A) = \frac{P(A | E_k)P(E_k)}{\sum_{i=1}^{n} P(A | E_i)P(E_i)} \]

Proof:

\[ P(E_k | A) = \frac{P(E_k \cap A)}{P(A)} = \frac{P(A | E_k)P(E_k)}{P(E_k)} = \frac{P(A | E_k)P(E_k)}{\sum_{i=1}^{n} P(A | E_i)P(E_i)} \]
Example:

Again suppose I have two coins.

    One is FAIR.

    The other has TWO heads.

If I randomly select a coin, flip it twice, and get heads both times, what is the probability I have the two headed coin?
“Concrete” Example

Three concrete companies deliver concrete for a big job

\[ E_i = \text{event concrete delivered from company } i \]
\[ P(E_i) = 70\%, 25\%, \text{ and } 5\% \quad \text{for } i = 1, 2, 3 \]

The probability a batch of concrete from each company fails a strength test is:

\[ P(F | E_i) = 0.09, 0.10, 0.20 \quad \text{for } i = 1, 2, 3 \]

• What is \( P(F) \)?
• What is \( P(E_i | F) \)?