LESSON 27: MATRIX SOLUTION TO DIRECT STIFFNESS METHOD

- SOLUTION TO NODE DISPLACEMENTS
- SOLUTION TO MEMBER FORCES
- Matrix Inverse.

A.) NODE DISPLACEMENTS:

LAST TIME WE FOUND:

$$\bar{q}_{global} = \bar{K}_{global} \cdot \bar{\delta}_{global}$$

JOINT FORCES ↗ $\bar{q}_{global}$; STIFFNESS ↖ $\bar{K}_{global}$; JOINT DISPLACEMENTS ↖ $\bar{\delta}_{global}$

BUT NOT ALL JOINTS ARE FREE TO DEFORM: (RESTRAINTS!)

We must subdivide matrices:

$$\begin{bmatrix} Q_f \\ \hline Q_s \end{bmatrix} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{bmatrix} \Delta_f \\ \hline \Delta_s \end{bmatrix}$$

Q_f = JOINT FORCES AT DOF'S FREE TO DISPLACE
Q_s = JOINT FORCES AT SUPPORTED DOF'S
Δ_f = UNKNOW JOINT DISPLACEMENTS
Δ_s = Support displacements ← 0, unless support settlement

$$\therefore \quad Q_f = K_{11}\Delta_f + K_{12}\Delta_s$$
$$Q_s = K_{21}\Delta_f + K_{22}\Delta_s$$

UNLESS SUPPORT SETTLEMENT,
$$\Delta_s = \bar{0}$$

$$\left.\begin{array}{l} Q_f = K_{11}\Delta_f \\ Q_s = K_{21}\Delta_f \end{array}\right\} \quad Q_f,\ K_{11},\ K_{21} \text{ are known.}$$

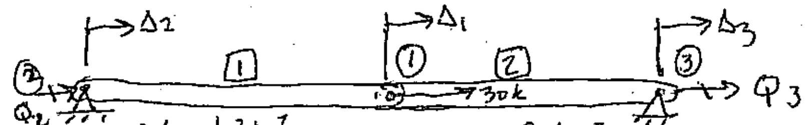
Need to solve for Δ_f

$$\Delta_f = K_{11}^{-1} Q_f$$

$$Q_s = K_{21} K_{11}^{-1} Q_f$$ ← FINDS SUPPORT Reactions

NOTES: 1) Deg. of det.? who cares! Same procedure
2) Need Matrix inverse (maybe)

EXAMPLE



$A_1 = 1.2\,in^2$ $\quad$ $A_2 = 0.6\,in^2$

$E_1 = 10{,}000$ ksi $\quad$ $E_2 = 20{,}000$ ksi

$L_1 = 120''$ $\quad$ $L_2 = 150''$

LOCAL STIFFNESS MATRICES

Member [1]:

$$\begin{array}{c} \\ ② \\ ① \end{array} \begin{array}{c} \begin{array}{cc} ② & ① \end{array} \\ \bar{K}_1 = \begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} \end{bmatrix} \end{array} = \frac{1.2(10{,}000)}{120}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

MEMBER [2]:

$$\begin{array}{c} \\ ① \\ ③ \end{array} \begin{array}{c} \begin{array}{cc} ① & ③ \end{array} \\ \bar{K}_2 = \begin{bmatrix} \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \end{array} = \frac{0.6(20000)}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 80\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} ① & ② & ③ \end{array} \\ \left[\begin{array}{c|cc} 100+80 & -100 & -80 \\ \hline -100 & 100 & 0 \\ -80 & 0 & 80 \end{array}\right] \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \quad \begin{matrix} \\ 0 \\ 0 \end{matrix}\Big\}\ \text{pinned!}$$

$Q_f = 30 \quad K_{11} = 180 \quad K_{12} = \begin{bmatrix} -100 & -80 \end{bmatrix} \quad \Delta_f = \Delta_1$

$Q_s = \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} \quad K_{21} = \begin{bmatrix} -100 \\ -80 \end{bmatrix} \quad K_{22} = \begin{bmatrix} 100 & 0 \\ 0 & 80 \end{bmatrix} \quad \Delta_s = \begin{bmatrix} \Delta_2 \\ \Delta_3 \end{bmatrix}$

$$Q_f = K_{11}\,\Delta_f \rightarrow \Delta_f = K_{11}^{-1}\,Q_f$$

where only 1 free DOF, scalar equ.

$$\Delta_f = \left(\frac{1}{180}\right)(30) = 1/6 \text{ in}$$

SOLVE FOR RXNS:

$$Q_s = K_{21}\,K_{11}^{-1}\,Q_f$$

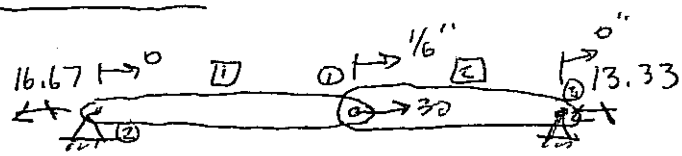
$$Q_s = \begin{bmatrix} -100 \\ -80 \end{bmatrix}\begin{bmatrix} \frac{1}{180} \end{bmatrix}\begin{bmatrix} 30 \end{bmatrix} = \begin{bmatrix} -100 \\ -80 \end{bmatrix}(1/6) = \begin{bmatrix} -16.67\,k \\ -13.33\,k \end{bmatrix}$$

$$Q_s = \begin{bmatrix} Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} -16.67\ k \\ -13.33\ k \end{bmatrix}$$

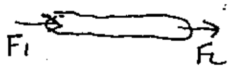
B.) MEMBER FORCES

NODE (a.k.a. joint) FORCES → MEMBER FORCES
(USE LOCAL STIFFNESS MATRIX)

<u>PRIOR EXAMPLE:</u>

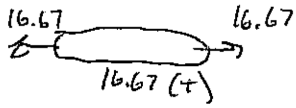


Member 1:
end forces

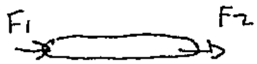


$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{bmatrix} 0 \\ 1/6'' \end{bmatrix}$$

$F_1 = -16.67\ k$
$F_2 = 16.67k$

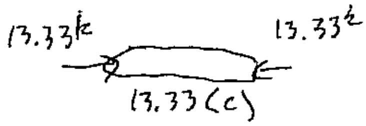


Member 2



$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 80 & -80 \\ -80 & 80 \end{bmatrix} \begin{bmatrix} 1/6 \\ 0 \end{bmatrix}$$

$F_1 = 13.33\ k$
$F_2 = -13.33k$



C.) MATRIX INVERSE (SQUARE MATRICES ONLY)

CONSIDER 3 CASES:
(THIS COURSE IS NOT A LINEAR ALGEBRA COURSE)

CASE 1) $\bar{A} \in \mathbb{R}^{1\times 1}$ (scalar) $\rightarrow \bar{A} = a_{11}$

$\bar{A}^{-1} = \frac{1}{a_{11}}$, matrix is a scalar, divide into 1.

CASE 2) $\bar{A} \in \mathbb{R}^{2\times 2} \rightarrow \bar{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\bar{A}^{-1} = \frac{\mathrm{adj}(\bar{A})}{\det(\bar{A})} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$\boxed{\det(\bar{A}) \neq 0}$

CASE 3) $\bar{A} \in \mathbb{R}^{n\times n}$
where $n > 2$

Refer to matrix algebra notes, use computational software package, etc.
(not on exam).

BONUS TOPIC
NUMERICAL ISSUES:

- NUMERICAL ISSUES WITH MATRIX INVERSE ARE COMMON
- A WELL PROGRAMMED SOLVER CAN OFTEN SIDESTEP THESE:

DANGEROUS APPROACH	SAFE APPROACH
CALCULATE K_{11}^{-1} (many methods) FIND $\Delta f = K_{11}^{-1} Q_f$	DIRECT SOLN. OF: $\Delta f = K_{11}^{-1} Q_f$ (GAUSS-JORDAN)

MAIN CAUSE OF NUMERICAL ISSUES IN STIFFNESS METHOD:

- MIXING VERY STIFF ELEMENTS w/ VERY FLEXIBLE ELEMENTS IN GLOBAL STIFFNESS MATRIX.

$\rightarrow$ POSSIBLE SOLUTIONS:
1) IGNORE EFFECTS OF COMPLIENT MEMBERS
2) MULTIPLE ANALYSES