LESSON 26: MEMBER AND GLOBAL STIFFNESS MATRICES

TOPICS:
- MATRIX REVIEW
- MEMBER STIFFNESS MATRICES
- GLOBAL STIFFNESS MATRIX

A) REVIEW OF MATRICIES

TENSORS: MULTIDIMENSIONAL ARRAYS WITH SPECIFIC RULES REGARDING MULTIPLICATION AND ADDITION: $\bar{A} \to a_{i,j,k}$

MATRICES: 2 DIMENSIONAL TENSORS : $\bar{A} \to a_{i,j}$

VECTORS: 1 DIMENSIONAL MATRICIES : $\bar{a} \to a_i$

SCALARS: SINGLE VALUES : a

- MATRIX SUBSCRIPTS: $a_{i,j}$

  i: row number, j: column number

$\bar{A} \in \mathbb{R}^{2\times3}$ means $\bar{A}$ is a matrix with 2 rows, 3 columns, populated by real numbers.

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- ADDITION
  - VECTOR ADDITION: $\bar{a} \in \mathbb{R}^{3\times1}$, $\bar{b} \in \mathbb{R}^{3\times1}$

$$\bar{a} + \bar{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$$ Legal

$\bar{a} \in \mathbb{R}^{3\times1}$, $\bar{b} \in \mathbb{R}^{1\times3}$

$$\bar{a} + \bar{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = ??$$ Not Legal

DIMENSIONS MUST AGREE

  - MATRIX ADDITION: $\bar{A} \in \mathbb{R}^{3\times3}$, $\bar{B} \in \mathbb{R}^{3\times3}$

$$\bar{A}+\bar{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

DIMENSIONS MUST AGREE

- MULTIPLICATIONS
  - MULTIPLICATION BY A SCALAR:

$$a \in \mathbb{R} \quad , \bar{b} \in \mathbb{R}^{3\times1} \quad , \bar{C} = \mathbb{R}^{3\times3}$$

$$a\bar{b} = a\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ ab_3 \end{bmatrix}$$

$$a\bar{C} = a\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} aC_{11} & aC_{21} & aC_{31} \\ aC_{12} & aC_{22} & aC_{32} \\ aC_{13} & aC_{23} & aC_{33} \end{bmatrix}$$

- Vector MULTIPLICATION:

$$\bar{a} \in \mathbb{R}^{3\times1} \quad , \quad \bar{b} \in \mathbb{R}^{1\times3}$$

MULTIPLY ROW X COLUMN

$$\bar{a}\,\bar{b} = \underset{3\times1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} a_1b_1 & a_2b_2 & a_3b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}}$$

ALSO KNOWN AS OUTER PRODUCT MATRIX

$$\bar{b}\bar{a} = \underset{1\times3}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \underset{1\times1}{\begin{bmatrix} b_1a_1 + b_2a_2 + b_3a_3 \end{bmatrix}} \text{ SCALAR}$$

INNER DIMENSIONS MUST AGREE

ALSO KNOWN AS INNER PRODUCT

- MATRIX MULTIPLICATION:

$$\bar{A} \in \mathbb{R}^{2\times3} \quad , \bar{B} \in \mathbb{R}^{3\times3}$$

$$\bar{A}\,\bar{B} = \underset{2\times3}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}$$

INNER DIMENSIONS MUST AGREE

$$\underset{2\times3}{\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \end{bmatrix}}$$

$$\underset{3\times3}{\bar{B}}\ \underset{2\times3}{\bar{A}}$$

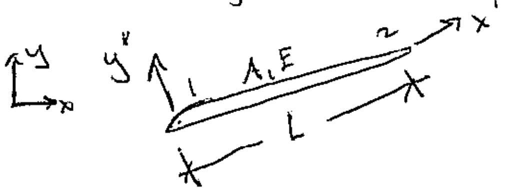
NOT EQUAL, NOT A LEGAL OPERATION.

$\bar{A}\bar{B} \neq \bar{B}\bar{A}$, Matrices do not commute

B) Member Stiffness Matrices

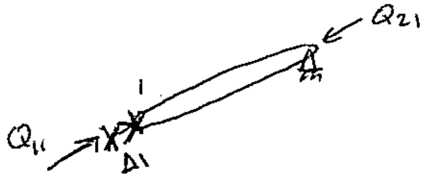
- Unique for every member:
- Based on local coordinates

→ Consider arbitrary truss bar:



x', y', Local coordinates

Consider displacement of joint 1

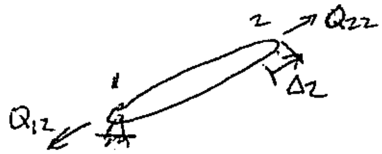


$$Q_{11} = \frac{AE}{L}\Delta_1$$

$$Q_{21} = -\frac{AE}{L}\Delta_1$$

negative x' direction

Consider displacement of joint 2



$$Q_{12} = \frac{-AE}{L}\Delta_2$$

$$Q_{22} = \frac{AE}{L}\Delta_2$$

Both at once: $Q_1 = Q_{11} + Q_{12} = \frac{AE}{L}(\Delta_1 - \Delta_2)$

$$Q_2 = Q_{21} + Q_{22} = \frac{AE}{L}(-\Delta_1 + \Delta_2)$$

Symmetry!

Put in matrix form: $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$

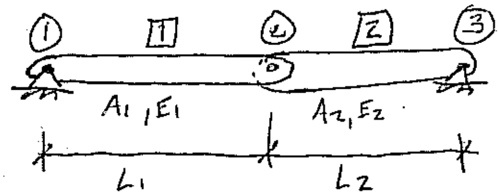
$$\bar{q} = \bar{K} \; \bar{\delta}$$

$\bar{K}$ = Member stiffness matrix for a truss bar

also written as $\bar{k} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

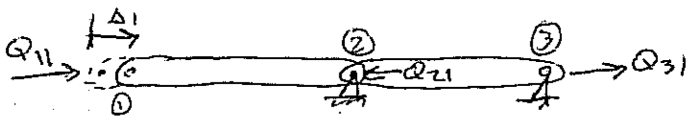
$$\bar{q} \in \mathbb{R}^{2\times 1}$$
$$\bar{k} \in \mathbb{R}^{2\times 2}$$
$$\bar{\delta} \in \mathbb{R}^{2\times 1}$$

C.) STRUCTURE STIFFNESS MATRIX



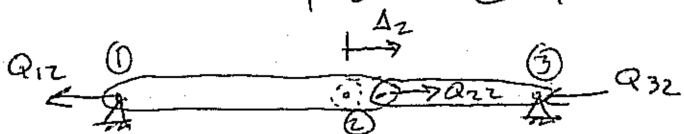
$$k_1 = \frac{A_1 E_1}{L_1}, \quad k_2 = \frac{A_2 E_2}{L_2}$$

- CAN DEFINE JOINT FORCES AT ①, ②, & ③
- CAN DEFINE JOINT DEFORMATIONS AT ①, ②, & ③



Deform ①, Lock ② & ③

$Q_{11} = k_1 \Delta_1$
$Q_{21} = -k_2 \Delta_1$
$Q_{31} = 0$



Deform ②, Lock ① & ③

$Q_{12} = -k_1 \Delta_2$
$Q_{22} = (k_1 + k_2)\Delta_2$
$Q_{32} = -k_2 \Delta_2$

Deform ③, Lock ① & ②

$Q_{13} = 0$
$Q_{23} = -k_2 \Delta_3$
$Q_{33} = k_2 \Delta_3$

$$
\begin{aligned}
Q_1 &= Q_{11} + Q_{12} + Q_{13} = k_1\Delta_1 \quad - k_1 \Delta_2 \\
Q_2 &= Q_{21} + Q_{22} + Q_{23} = -k_1\Delta_1 \quad (k_1+k_2)\Delta_2 \quad -k_2\Delta_3 \\
Q_3 &= Q_{31} + Q_{32} + Q_{33} = \quad -k_2\Delta_2 \quad k_2\Delta_3
\end{aligned}
$$

MATRIX NOTATION:

$$
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix}
$$

$$\bar{q}_{global} = \bar{K}_{global} \cdot \bar{\delta}_{global}$$

$\bar{q}_{global} \in \mathbb{R}^{num\ joints \times 1}$, $\bar{K}_{global} \in \mathbb{R}^{num.joints \times num.joints}$

$\bar{\delta}_{global} \in \mathbb{R}^{num\ joints \times 1}$

We can also assemble global stiffness matrix from member stiffness matrices.

force → ① ② ← INDICATES JOINT #

Consider
$$\bar{K}_1 = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

disp.

$$\bar{K}_2 = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Plug into appropriate slots in $\bar{K}_{global}$

$$\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} k_1 & -k_1 & \\ -k_1 & k_1+k_2 & -k_2 \\ & -k_2 & k_2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

RAPID / ACCURATE SORTING OF THESE COEFFICIENTS IS COMPUTERS' MAIN ADVANTAGE IN APPLYING THIS METHOD, PARTICULARLY FOR LARGE STRUCTURES WITH MANY MEMBERS