1. Mathematical models are used to predict the growth of a population, i.e. population size at some future date. The simplest model is that for exponential growth. The calculation requires a knowledge of the organism's maximum specific growth rate ($\mu_{\text{max}}$). A value for this coefficient can be obtained from field observations of population size or from laboratory experiments where population size is monitored as a function of time:

<table>
<thead>
<tr>
<th>Time (d)</th>
<th>Biomass (mg/L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>68</td>
</tr>
<tr>
<td>2</td>
<td>91</td>
</tr>
<tr>
<td>3</td>
<td>123</td>
</tr>
<tr>
<td>4</td>
<td>166</td>
</tr>
<tr>
<td>5</td>
<td>224</td>
</tr>
</tbody>
</table>

Calculate $\mu_{\text{max}}$ for this population assuming exponential growth; include appropriate units. (0.3d\(^{-1}\))

**SOLUTION:**
There are different ways one could solve this problem, but there is one method that is recommended. E.g., one could work with the differential form of the equation:

$$\frac{dX}{dt} = \mu_{\text{max}} X$$

and calculate $\Delta X$ for each interval $\Delta t$ and then divide these by the “average” value of $X$ for each time interval to arrive at 5 estimates of $\mu_{\text{max}}$. The result would be as shown in the table below.

<table>
<thead>
<tr>
<th>Time (d)</th>
<th>Biomass (mg/L)</th>
<th>$\Delta X$ (mg/L)</th>
<th>$\Delta t$ (d)</th>
<th>$\Delta X/\Delta t$ (mg/L-d)</th>
<th>Avg.X</th>
<th>$\mu_{\text{max}}$ (1/d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>68</td>
<td>18</td>
<td>1</td>
<td>18</td>
<td>59</td>
<td>0.298</td>
</tr>
<tr>
<td>2</td>
<td>91</td>
<td>23</td>
<td>1</td>
<td>23</td>
<td>70</td>
<td>0.298</td>
</tr>
<tr>
<td>3</td>
<td>123</td>
<td>32</td>
<td>1</td>
<td>32</td>
<td>107</td>
<td>0.298</td>
</tr>
<tr>
<td>4</td>
<td>166</td>
<td>43</td>
<td>1</td>
<td>43</td>
<td>147</td>
<td>0.298</td>
</tr>
<tr>
<td>5</td>
<td>224</td>
<td>58</td>
<td>1</td>
<td>58</td>
<td>195</td>
<td>0.298</td>
</tr>
</tbody>
</table>
Alternatively, one can work with the integrated form of the rate law:

\[ X_t = X_0 \exp(\mu_{\max} t) \]

\[ \ln(X_t) = \ln(X_0) + \mu_{\max} \cdot t \]

Now if \( \ln(X) \) is regressed against \( t \), the slope will be \( \mu_{\max} \). This is the recommended method of solution. The value of \( \mu_{\max} \) calculated in this fashion is found to be 0.30 \( d^{-1} \).

2. Once a value for \( \mu_{\max} \) has been obtained, the model may be used to project population size at a future time. Assuming that exponential growth is sustained, what will the population size in Problem #1 be after 25 days? \( 90,402 \text{ mg/L} \)

**SOLUTION:**

The integrated rate law may be used to calculate \( X_{t=25d} \) as:

\[ X_{t=25} = X_0 \exp(\mu_{\max} t) = 50 \cdot \exp(0.3 \cdot 25d) = 9.0 \times 10^4 \]

The units for \( X \) are mg/L.

3. Exponential growth cannot be sustained forever because of constraints placed on the organism by its environment, i.e. the system's carrying capacity. This phenomenon is described using the logistic growth model. Calculate the size of the population in Problem #1 after 25 days, assuming that logistic growth is followed and that the carrying capacity is 50,000 mg/L. What percentage of the exponentially-growing population size would this be? \( 32,205 \text{ mg/L; 35.6\%} \)

**SOLUTION:**

The integrated form of the logistic equation is:

\[ X_t = \frac{K}{1 + \left( \frac{K - X_0}{X_0} \right) \exp(-\mu_{\max} t)} \]

\[ X_t = \frac{50,000 \text{ mg}}{1 + \left( \frac{50,000 - 50}{50} \right) \exp(-0.3 \cdot 25d)} = 32,205 \]

This \( 32,205 \text{ population is only } (32,205/90,000) 35.6\% \text{ of the population predicted with the exponential model.} \)

4. Food limitation of population growth is described using the Monod model. Population growth is characterized by the maximum specific growth rate (\( \mu_{\max} \)) and the half-saturation constant for growth (\( K_s \)). Take the population vs. time data in Problem #1 and calculate the maximum specific growth rate that would be required using the Monod model with a substrate concentration of 25 mg/L and a \( K_s \) of 5 mg/L. What percentage of the growth rate (\( \mu \)) for an exponentially-growing population would this be? Explain why the growth rate for the Monod model is higher or lower than the growth rate for the exponential model. \( 0.36 \text{ d}^{-1}; 120\% \)

**SOLUTION:**
In the Monod model, the growth rate equals the maximum potential growth rate times the ratio of food to (food + half-saturation constant). Since we know that the growth rate for this set of data is 0.3 d\(^{-1}\), we know that:

\[
0.3d^{-1} = \mu = \mu_{\text{max}} \left( \frac{S}{K_s + S} \right) = \mu_{\text{max}} \left( \frac{25 \text{ mg}} {L} \right) = 0.83 \cdot \mu_{\text{max}}
\]

\[
\therefore \mu_{\text{max}} = \frac{\mu}{0.83} = 0.36d^{-1}
\]

This value is 120% higher than the value for the exponential model. The growth rate (dX/dt) predicted by the two models is identical because it must fit the same set of data. In the Monod model, the maximum potential growth rate is multiplied by a factor (S/S+K_s) that is always less than one. Thus, the maximum potential growth rate in the Monod model must be larger than the intrinsic growth rate used in the exponential model.

5. The two coefficients defined in Problem #4 (\(\mu_{\text{max}}\) and \(K_s\)) describe the organism's ability to function in the environment. Populations with a high \(\mu_{\text{max}}\) grow rapidly and take up substrate very quickly. Those with a low \(K_s\) are able to take up substrate quite efficiently, reducing it to low levels. These characteristics are important when considering the use of microorganisms to clean up pollution from potentially toxic chemicals. Consider two genetically engineered organisms intended for use in a chemical spill cleanup. Organism "A" has a \(\mu_{\text{max}}\) of 1 d\(^{-1}\) and a \(K_s\) of 0.1 mg/L. Organism "B" has a \(\mu_{\text{max}}\) of 5 d\(^{-1}\) and a \(K_s\) of 5 mg/L. Chemical levels are initially on the order of 100 mg/L; the goal is to reduce concentrations to below 0.1 mg/L. We wish to use the organisms in sequence - first one organism to rapidly reduce chemical levels before they can spread and second, an organism to reduce chemical levels to the target concentration of 0.1 mg/L. Which organism ("A" or "B") would be most effective in rapidly reducing levels of pollution? Which organism ("A" or "B") would be most effective in reducing the pollutant to trace levels? Back your answer up with calculations. (B is effective in achieving fast reductions; A reduces pollutants to lower levels)

ANSWER:
The organism that has a fast initial growth rate would be best at rapidly reducing the level of pollution. The initial growth rates are:

A: \(\mu = \mu_{\text{max}}(S/K_s + S) = 1(\text{d}^{-1}) \times (100/(100+0.1)) = 1 \text{ d}^{-1}\)

B: \(\mu = \mu_{\text{max}}(S/K_s + S) = 5(\text{d}^{-1}) \times (100/(100+5)) = 4.76 \text{ d}^{-1}\)

Therefore B is most effective in achieving rapid reductions in concentration. However, at low concentrations (~1 mg/L), the growth rates are reversed.

A: \(\mu = \mu_{\text{max}}(S/K_s + S) = 1(\text{d}^{-1}) \times (1/(1+0.1)) = 0.9 \text{ d}^{-1}\)

B: \(\mu = \mu_{\text{max}}(S/K_s + S) = 5(\text{d}^{-1}) \times (1/(1+5)) = 0.8 \text{ d}^{-1}\)

Since substrate uptake rate is \(-1/Y\) times the bacterial growth rate, organism A will now be better at growing and reducing the substrate concentration.

6. In wastewater treatment, organism biomass increases as pollutants are taken up and metabolized. This increase is reflected in the amount of sludge generated at the wastewater treatment plant, a residue that must receive safe disposal. Engineers use the yield coefficient (Y) to calculate biomass
(sludge) production. Laboratory studies have shown that microorganisms produce 1.0 mg/L of biomass in reducing the concentration of a pollutant by 5.0 mg/L. Calculate the yield coefficient, specifying the units of expression. (0.2 mg biomass / mg substrate)

**ANSWER:**
The yield coefficient is the amount of biomass produced per unit of substrate used. Here, this would be 1 mg/L biomass/5.0 mg/L food = 0.2 mg biomass/mg substrate.

7. When food supplies have been exhausted, populations die away. This exponential decay is described by a simple modification of the exponential growth model. Engineers use this model to calculate the length of time for which a swimming beach must remain closed following pollution with fecal material. For a population of bacteria with an initial biomass of 90 mg/L and a $k_d = 0.35 \, \text{d}^{-1}$, calculate the time necessary to reduce the population size to 15 mg/L. (5.1 d)

**SOLUTION:**
The governing equation is:
$$X = X_0 \exp(-k_d t)$$
Using the given values and solving for $t$,
$$15 \, \text{mg/L} = 90 \, \text{mg/L} \times \exp(-0.35 \, \text{d}^{-1} \times t)$$
$$t = -\frac{1}{k_d} \ln\left(\frac{X}{X_0}\right) = -\frac{1}{0.35} \times \ln\left(\frac{15}{90}\right) = 5.1 \, \text{d}$$