Systems Design and Engineering

Success Runs, Optimal Replacement & Repairman Problem

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Success Runs

Consider the probability transition matrix:

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 & \dots \\ p_1 & r_1 & q_1 & 0 & 0 & \dots \\ p_2 & 0 & r_2 & q_2 & 0 & \dots \\ p_3 & 0 & 0 & r_3 & q_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where $p_i > 0$, $q_i > 0$ and $p_i + q_i + r_i = 1$ for $i = 0, 1, 2....$ Consider a situation where a component of a system (a component of a computer, a pipe network segment etc.) has an active life, measured in discrete units, that is given by the random variable T, where $Pr[T = k] = a_k$, for $k = 1, 2...$, where $\sum_k = 1^\infty a_k = 1$. Suppose one starts with a fresh component, and each component is replaced by a new component upon failure (you can also consider a replacement as a service intervention).

Consider the above matrix in the context of such a success run Markov chain, where $r_k = 0$. The age of the component reverts to 0 on failure and given that the age of the component in service is currently k, the failure occurs in the next time period with the following conditional probability:

$$p_k = \frac{a_{k+1}}{a_{k+1} + a_{k+2} + \dots} \tag{1}$$

Therefore q_k is the probability of the component not failing and instead living another day, so $q_k = 1 - p_k$.

Age Replacement Policy

Following from the previous section, let X_n be the age of the component in service at time n. Hence at time of failure X_n is set to 0 by definition. Consider the policy that calls for replacement of the component at age N or when it fails, which ever occurs first. Then X_n has a success run Markov chain with following $N \times N$ probability transition matrix (state space is defined on the set of integers from $0 - (N - 1)$):

$$\mathbf{P} = \begin{bmatrix} p_0 & 1-p_0 & 0 & 0 & 0 & \dots & 0 \\ p_1 & 0 & 1-p_1 & 0 & 0 & \dots & 0 \\ p_2 & 0 & 0 & 1-p_2 & 0 & \dots & 0 \\ p_3 & 0 & 0 & 0 & 1-p_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As state 0 corresponds to a new unit π_0 is the long run probability of replacement during any time unit or the long run replacement per unit time. At $X_n = N - 1$ a planned replacement occurs. Therefore, π_{N-1} is the long run planned replacement per unit time. Therefore rate of failure of service is the difference between π_0 and π_{N-1}. Solving the following system of equations for the above matrix:

$$\begin{aligned} p_0\pi_0 + p_1\pi_1 + \dots \quad + p_{N-2}\pi_{N-2} + \pi_{N-1} &= \pi_0 \quad (2)\\ (1-p_0)\pi_0 &= \pi_1 \quad (3)\\ (1-p_1)\pi_1 &= \pi_2 \quad (4)\\ (1-p_{N-2})\pi_{N-2} &= \pi_{N-1} \quad (5)\\ \pi_0 + \pi_1 + \dots \quad + \pi_{N-2} + \pi_{N-1} &= 1 \quad (6) \end{aligned}$$

When solved this leads to:

$$\begin{aligned} \pi_0 = & \quad \tfrac{1}{A_1+A_2+A_3\ldots+A_n} && (7)\\ where \quad 1-p_k = & \quad \frac{A_{k+2}}{A_{k+1}} && (8)\\ and \quad \pi_{N-1} = & \quad A_N\pi_0 && (9) \end{aligned}$$

Therefore if the cost of replacement is C and an additional cost of K is incurred when a failure in service occurs, then the long run total cost per unit time is $C\pi_0 + K(\pi_0 - \pi_{N-1}$. Choose the replacement age N to minimize this cost. What is the mean time between replacements?

Equivalence Property

The steady state output of an $M/M/s$ service facility, where $s\mu > \lambda$, is also a Poisson process with parameter λ.

Applications: Repairman Model

A system is composed of N machines of which at most $M \leq N$ can be operating at one time. The rest are spares. When a machine is operating it operates a random length of time until failure with parameter μ. When a machine fails it undergoes repair. At most R machines are in repair at any point of time. The repair time is exponentially distributed with parameter λ. Hence a machine cane be in any of the four states:

- Operating
- Up but not operating
- In repair
- Waiting for repair

There are a total of N machines in the system. At most M can be operating. At most R can be in repair. Let $X(t)$ be a random variable denoting the number of up machines at the time t. Hence, we can say:

- Number of machines operating: $min\{X(t), M\}$
- Number of spares: $max\{0, X(t) - M\}$
- Number of down machines: $Y(t) = N - X(t)$
- Number in repair: $max\{0, Y(t) - R\}$

$X(t) = n$ is a finite state birth and death process with the parameters:

$$\lambda_n = \lambda \times min\{N-n, R\} = \begin{cases} \lambda R \quad for\ n = 0, 1, ..., N-R, \\ \lambda(N-n) \quad for\ n = N-R+1, ..., N \end{cases}$$

and

$$\mu_n = \mu \times min\{n, M\} = \begin{cases} \mu n \quad for\ n = 0, 1, ..., M, \\ \mu M \quad for\ n = M+1, ..., N \end{cases}$$

All else can be routinely discovered.