Systems Design and Engineering

Problem sets

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Problem 1

A Markov chain $X_0, X_1, X_2...$ has the transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

Every period that the process spends in state 0 it incurs a cost of \$2. Every period that the process spends in state 1 it incurs a cost of \$5. Every period that the process spends in state 2 it incurs a cost of \$3. What is the long run cost per period associated with this Markov chain?

Problem 2

Consider the following directed graph. If the node 6 is an absorbing node, then estimate the expected number of steps that will be necessary for a random walker to land up at node 6 given that he may start at any of the other possible nodes.

Problem 3

An airline system has 2 computers only one of which is in operation at any given time. A computer can break down on any given day with probability p. There is a single repair facility that takes 2 days to repair the machine. Only one computer at a time ca be repaired and both machines are extremely unlikely to fail on the same day. Form a Markov chain by taking as states the pairs (x, y) where x is the number of machines in operation at the end of the day, and $y = 1$ if a day's work has been expended on a machine not yet repaired and 0 otherwise. Calculate the reliability of the system defined by the percentage of the time that at least one computer is functioning.

Problem 4

Consider the probability transition matrix:

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 & \dots \\ p_1 & r_1 & q_1 & 0 & 0 & \dots \\ p_2 & 0 & r_2 & q_2 & 0 & \dots \\ p_3 & 0 & 0 & r_3 & q_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where $p_i > 0$, $q_i > 0$ and $p_i + q_i + r_i = 1$ for $i = 0, 1, 2....$ Consider a situation where a component of a system (a component of a computer, a pipe network segment etc.) has an active life, measured in discrete units, that is given by the random variable T, where $Pr[T = k] = a_k$, for $k = 1, 2....$ Suppose one starts with a fresh component, and each component is replaced by a new component upon failure (you can also consider a replacement as a service intervention). Let X_n be the age of the component in service at time n.

Consider the above matrix in the context of a success run where $p_n = \beta$, $r_n = 0$ and $q_n = \alpha$ where α is the probability of failure and $\beta = 1 - \alpha$.

$$\beta = \quad p_n = \quad \frac{a_{n+1}}{a_{n+1} + a_{n+2} + ...} \tag{1}$$

Consider the long run behavior of this Markov chain and how it can be used to model optimal age replacement policies.