Systems Design and Engineering

Long Run Behavior of Markov Chains

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We are particularly interested in estimating how Markov chain or process behaves after a long period of time has passed (or after multiple steps). Specifically the goal is to identify a steady state behavior of the system if any. Relevant concepts we discuss are: *limiting distributions and stationary distributions.*

For a discrete time Markov chain $\{X_m\}$ with probability transition matrix $\mathbf{P}$ consider the following equation:

$$\lim_{m\to\infty} p_{ij}^{(m)} = \pi_j > 0 \quad \forall i \;\&j = 0, 1, 2...N; \tag{1}$$

It implies that after a long time the probability that the system is in the state j given that it started in state i is independent of the starting state i. It also means that $\mathbf{P}^m$ approaches a limiting behavior as m goes to infinity. In other words, all the rows in the m-step transition matrix becomes identical. The quantities $\{\pi_j\}$ are referred to as the *limiting or steady state probabilities* of the Markov chain. Hence, the limiting distribution is given by the vector:

$$\overrightarrow{\pi} = \{\pi_0, \pi_1, ..., \pi_N\} \tag{2}$$

Consider the example of the matrix:

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

As illustrated in class - for the condition $0 < a, b < 1$, the limiting distribution for the above system reduces to $\overrightarrow{\pi} = \{\frac{b}{a+b}, \frac{a}{a+b}\}$.

Calculating Steady State Probabilities

When the limiting probabilities exist they can be calculated as follows:

$$\lim_{m\to\infty} \overrightarrow{\pi}^{(m)} = \lim_{m\to\infty} \overrightarrow{\pi}^{(m-1)}\mathbf{P} \tag{3}$$

Letting $\overrightarrow{\pi} = \{\pi_0, \pi_1, ..., \pi_N\}$ represent the limiting probability vector -

$$\lim_{m\to\infty} \overrightarrow{\pi}^{(m)} = \lim_{m\to\infty} \overrightarrow{\pi}^{(m-1)} = \overrightarrow{\pi} \tag{4}$$

Therefore we can say:

$$\overrightarrow{\pi} = \overrightarrow{\pi}\mathbf{P}, \quad or \tag{5}$$

$$\mathbf{0} = \overrightarrow{\pi}\mathbf{Q} \tag{6}$$

This leads to the system of equations given by:

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}; \quad j = 0, 1, ...N, \tag{7}$$

given the condition that:

$$\sum_{k=0}^{N} \pi_k = 1 \tag{8}$$

The above system of equations are referred to as the *stationary equations* of a Markov chain.

Stationary Probability Distribution

The solution to equations (7,8) is referred to as the *stationary distribution.* When the limits defined in equation (1) exists, and there is a solution to the system of stationary equations then the system shows long term limiting behavior and the limiting distribution is the same as the stationary distribution.

However, if the limit in equation (1) does not exist for the system then the solution to the system of equations presents a stationary distribution and no limiting distribution exists. In such cases the probability $\{\pi_j\}$ reflects the percentage of the time that the system spends in the state j.

Consider the following systems:

- The first system alternates between two states a and b and its transition matrix is given as:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  The solution to the stationary equations exist and is $(\pi_a, \pi_b) = (1/2, 1/2)$. However, there is no limiting distribution as the limit in this system does not exist. In the long run the system keeps oscillating between the states a and b effectively spending 50% of its time in each.

- Now consider the system:

$$\mathbf{P} = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

  The solution to the stationary equations still exist and is $(\pi_a, \pi_b) = (1/2, 1/2)$. Indeed successive multiplication of $\mathbf{P}$ shows that $\mathbf{P}^{(m)}$ does indeed converge to a matrix all of whose entries are 1/2.

Find the stationary distribution (π_0, π_1, π_2) for the following system:

$$\mathbf{P} = \begin{bmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{bmatrix}$$