SSE 2300: Service Systems Design and Engineering

Birth and Death Process

February 25, 2009

Definitions

A **birth** refers to the arrival of a new customer and **death** refers to the departure of a served customer.

State of the system at time t, $(t \geq 0)$ is given by $N(t)$= the number of customers in the system at time t.

Individual births and deaths occur randomly and the mean occurrence rates depend only upon the current state of the system.

Assumptions

The following assumptions apply:

- Given $N(t) = n$, the current probability distribution of the remaining time until the next birth (arrival) is exponential with parameter λ_n $(n = 0, 1, 2, ...)$.
- Given $N(t) = n$, the current probability distribution of the remaining time until the next death (service completion) is exponential with parameter μ_n $(n = 0, 1, 2, ...)$.
- The random variable of assumption 1 (the remaining time until the next birth) and the random variable of assumption 2 (the remaining time until the next death) are mutually independent. The next transition in the state of the process is either $n \to n + 1$ (a single birth) or $n \to n - 1$ (a single death) depending on whether the former or latter random variable is smaller.

λ_n and μ_n can be different for different values of n if arriving customers become increasingly likely to *balk* as n increases, and *renege* as queue size increases respectively.

Balance Equation

$E_n(t)$= number of times the process enters state n by time t.
$L_n(t)$= number of times the process leaves state n by time t.

Mean rate at which process enters state n is given by:

$$\lim_{t\to\infty} \frac{E_n(t)}{t} \tag{1}$$

Mean rate at which process leaves state n is given by:

$$\lim_{t\to\infty} \frac{L_n(t)}{t} \tag{2}$$

Rate in = Rate out. For any state of the system n, mean entering rate = mean leaving rate.

Relevant Formulae

$$P_n = C_n.P_0 \tag{3}$$

Given that:

$$\sum_{n=0}^{\infty} P_n = 1 \tag{4}$$

Where:

$$C_n = \frac{\lambda_{n-1}\lambda_{n-2}...\lambda_0}{\mu_n\mu_{n-1}...\mu_1} \tag{5}$$

The following formula apply, where the symbols have their usual meaning:

$$L = \sum_{n=0}^{\infty} nP_n \tag{6}$$

$$L_q = \sum_{n=0}^{\infty}(n-s)P_n \tag{7}$$

Consider the M/M/1 case:

$$\lambda_n = \lambda, n = 0, 1... \tag{8}$$

$$\mu_n = \mu, n = 0, 1... \tag{9}$$

$$C_n = (\frac{\lambda}{\mu})^n = \rho^n \tag{10}$$

$$P_0 = (1-\rho) \tag{11}$$

$$P_n = (1-\rho)\rho^n \tag{12}$$

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\lambda}{\mu-\lambda} \tag{13}$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} \tag{14}$$

Consider the M/M/s case:

$$\lambda_n = \lambda, n = 0, 1... \tag{15}$$

$$\mu_n = \begin{cases} n\mu, \ n \leq s \\ s\mu, \ n > s \end{cases} \tag{16}$$

$$C_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} \ for \ n = 1, 2, ...s \\ \frac{(\lambda/\mu)^n}{s!s^{(n-s)}} \ for \ n = s, s+1, ... \end{cases} \tag{17}$$

$$L_q = \frac{P_0(\lambda/\mu)^s\rho}{s!(1-\rho)^2} \tag{18}$$

$$P_0 = 1/[\sum_{n=0}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\frac{1}{1-\lambda/(s\mu)}] \tag{19}$$

Consider the finite queue variation of the M/M/1 model (M/M/1/K)

$$\lambda_n = \begin{cases} \lambda \ for \ n = 0, 1, 2, ...K-1 \\ 0 \ for \ n \geq K \end{cases}$$

$$C_n = \begin{cases} (\frac{\lambda}{\mu})^n = \rho^n \ for \ n = 0, 1, 2, ...K \\ 0 \ for \ n = 0, 1, 2, ...K - 1 \end{cases}$$

$$P_0 = \frac{1 - \rho}{1 - \rho^{K+1}} \tag{20}$$

$$P_n = \frac{1 - \rho}{1 - \rho^{K+1}} \rho^n, \ for \ n = 0, 1, 2, ...K \tag{21}$$

$$L = \frac{\rho}{1 - \rho} - \frac{(K + 1)\rho^{K+1}}{1 - \rho^{K+1}} \tag{22}$$

$$L_q = L - (1 - P_0) \tag{23}$$