CE 5690 - Descriptive Modeling of Data

Review of Probability Theory

Sample point x ⊂ Event E ⊂ Sample Space S

Probability -- how we describe the likelihood of different events
-- the language of uncertainty

Experiment – any process whose outcome is subject to uncertainty

Sample Space S -- set of all possible outcomes

Sample point x -- one single outcome $x \in S$

Event E -- a subset of S $E \subset S$

Simple Events -- a set containing a single sample point $E = \{x_0\}$

Compound Events -- a set containing more than one sample point

$E = \{1,2,3\}$ or $E[0,1]$ ← all #'s on interval 0,1

Ex. Rolling a die

$S = \{1,2,3,4,5,6\}$

$x = 3$

$E = \{3\}$

$E = \{1,2,3\}$

Review of Set Theory

Let A and B be two events in the sample space S.

Set Operations:

Union (U) Set of all outcomes that belong to either A or B (or both)

$A \cup B$

Intersection (∩) Set of outcomes in S that belong to both A and B.

$A \cap B$

Complement (') $A' \Rightarrow$ Set of outcomes in S that DO NOT belong to A.

A' or A^c

Mutually exclusive or disjoint events; empty set ϕ

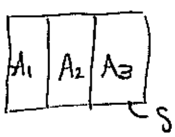
A and B have no outcomes in common

A and B are mutually exclusive iff $A \cap B = \emptyset$

Collectively Exhaustive

Suppose $A_1, A_2 \ldots A_n$ are n events in S.

If $\cup A_i = S$, then these n events are collectively exhaustive



$A_1 \cup A_2 \cup A_3 = S$

Ex. Roll a die

$A = \{2,4,6\}$ all even rolls

$B = \{4,5,6\}$ roll of at least 4

$A \cup B = \{2,4,5,6\}$

$A \cap B = \{4,6\}$

$A' = \{1,3,5\}$

Are A and B mutually exclusive?
NO.

Ex. $A \cap A^c = ?$ $\emptyset$

$A \cup A^c = ?$ S

Axioms for Probability

P(·) is a function that maps subsets of $\mathcal{S}$ into [0,1]
- assign a probability to every outcome in S
- mapping of an event to a real number

1) For every event A, $P(A) \geq 0$.

2) For the sample space $\mathcal{S}$, $P(\mathcal{S}) = 1$.

3) If $\{ A_i \mid i = 1, \ldots, n \}$ is a finite collection of *mutually exclusive events*, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma\, P(A_i)$$

$\Rightarrow$ If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Properties of Probability that follow from Axioms

i) $P(A) \leq 1$

② Because $P(A^c) \geq 0$ by axiom 1.
$P(A) \leq 1$ since $P(A) = 1 - P(A^c)$

ii) For every event A, $P(A) = 1 - P(A')$

① $S = A \cup A^c \therefore P(A \cup A^c) = P(S) = 1$
Axiom 3 $\longrightarrow P(A \cup A^c) = P(A) + P(A^c)$ $\Big\}$ $P(A) = 1 - P(A^c)$

iii) $P(\phi) = 0$ where ϕ is the empty set.
$S = S \cup \emptyset$ ~~and $S \cap \emptyset = \emptyset$~~ and S and $\emptyset$ are disjoint
$1 = P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$

iv) For mutually exclusive events A and B, $P(A \cap B) = 0$.

$A \cap B = \emptyset \therefore P(A \cap B) = P(\emptyset) = 0$

v) For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

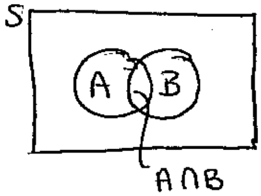
What about three events? A, B, C?
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Conditional Probability: P(A | B) = Probability of A given that B occurred.

Definition: $P(A \mid B) = P(A \cap B) / P(B)$

Result: $P(A \cap B) = P(A \mid B) * P(B)$

* Revision of probability assignment given new information
  $P(A)$ = prior probability of A
  $P(A|B)$ = posterior probability of A. given B already occurred



→ Reduction of sample space because know B occurred.

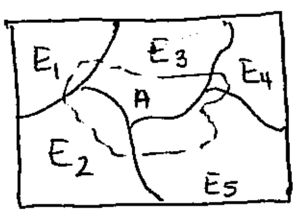
$P(A|B) = \frac{P(A \cap B)}{P(B)}$ ⇒ portion of outcomes in B which are also contained in A.

Law of Total Probability

Let $E_1, E_2, \cdots, E_n$ be a partition of the sample space S.

Note: $\{E_1, E_2, \cdots, E_n\}$ is a partition of the sample space S if they are *mutually exclusive & collectively exhaustive.*
i.e. every point in S is contained in one and only one E_i.

Then for any event A, $P(A) = \sum_{i=1}^{n} P(A \mid E_i) P(E_i)$



Proof: $A = \bigcup_{i=1}^{n} [A \cap E_i]$

↓ E_i's disjoint

$\therefore P(A) = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(A|E_i) P(E_i)$

conditional prob

Bayes Theorem
Let $E_1, E_2, \cdots, E_n$ be a partition of the sample space S
with $P(E_i) > 0$ for $i = 1, \ldots, n$.

Then for any event A for which P(A) > 0: $P(E_k \mid A) = \dfrac{P(A \mid E_k)P(E_k)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}$

Proof: $P(E_k \mid A) = \dfrac{P(E_k \cap A)}{P(A)}$ ← conditional probability

$$= \frac{P(A \mid E_k)\, P(E_k)}{P(A)}$$

$$= \frac{P(A \mid E_k)\, P(E_k)}{\sum_{i=1}^{n} P(A \mid E_i)\, P(E_i)}$$ ← Law of Total Probability

Example. Suppose I have two coins. 1 is Fair, the other has two heads.
(a) If I randomly select a coin and flip it twice, What is the probability I get heads both times?

Use law of total probability
Let E_1 = Fair Coin Selected ; E_2 = Two headed coin selected

$$P(HH) = \sum_{i=1}^{2} P(HH \mid E_i)\, P(E_i) = P(HH \mid E_1)\, P(E_1) + P(HH \mid E_2)\, P(E_2)$$

$$= (1/4)(1/2) + (1)(1/2) = \underline{\underline{5/8}}$$

Given flipped HH,
(b) What is probability selected two headed coin?

$$P(E_2|HH) = \frac{P(HH|E_2)P(E_2)}{\sum P(HH|E_i)P(E_i)}$$

$$= \frac{P(HH|E_2)P(E_2)}{P(HH)}$$

$$= \frac{1\,(1/2)}{5/8} = \underline{\underline{4/5}}$$

Independence

Definition of Independence: When knowledge of one event A has no impact on the probability of another B. $A \perp B$ ie. roll of one die does not influence another.

In mathematical notation, events A and B $\subset$ S are independent if and only if

$$P(B \mid A) = P(B)$$

Theorem

If events A and B are independent then

i) $P(B|A) = P(B)$ → definition

ii) $P(A|B) = P(A)$

By definition of independence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \overset{\text{Cond. Prob.}}{=} \frac{P(B|A)\,P(A)}{P(B)} = \frac{P(B)\,P(A)}{P(B)} = P(A)$$

iii) $P(A \cap B) = P(A)\,P(B)$

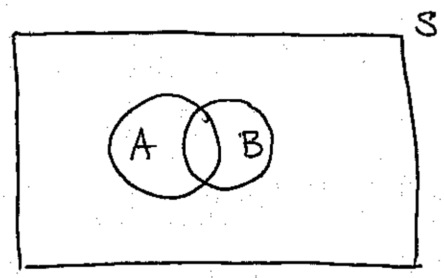
$$P(A \cap B) = P(A|B)\,P(B)$$
$$= P(A)P(B).$$

Q ↳ If $A \perp B$, can A and B also be disjoint?

· Conditional Probability

$P(A|B)$ = Probability of A given that B occurred.

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{or} \quad P(A \cap B) = P(A|B) * P(B)$$

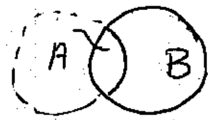


$P(S) = 1.$

← Samplespace has several outcomes including Events A and B.

↳ If B occurs, samplespace is reduced

⇓



$P(B|B) = 1.$

↳ revise probability assigned to A in light of new information (i.e. B occurred)

$P(A)$ - prior probability of A
- probability A will occur relative to all outcomes in S

$$P(A) = P(A|S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ ⇒ Portion of outcomes in B which are also contained in A.

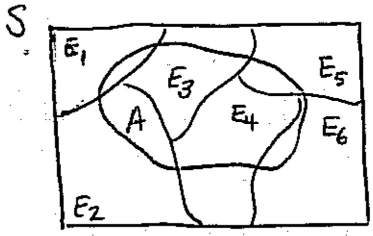
Just pose questions

Ex. Suppose I have two coins in my pocket. One is fair and the other has two heads.

Q. If I randomly select a coin and flip it twice, what is the probability I get heads on both flips?

Q. Now, suppose I select a coin and see it is the fair coin, what is the probability I get heads on both flips?

Law of Total Probability



← Partition sample space with mutually exclusive + collectively exhaustive events.

For any event A, $P(A) = \sum_{i=1}^{n} P(A|E_i)\,P(E_i)$

Proof: $A = \bigcup_{i=1}^{n} [A \cap E_i]$

$\downarrow$ E_i's are disjoint

$$\therefore\; P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

conditional probability

$$= \sum_{i=1}^{n} P(A|E_i)\,P(E_i)$$

$\boxed{\rightarrow \text{Ex. Part a}}$

Bayes Theorem

$E_1, E_2, \ldots E_n$ is a partition of S.

For any event A, $$P(E_k|A) = \frac{P(A|E_k)\,P(E_k)}{\sum_{i=1}^{n} P(A|E_i)\,P(E_i)}$$

Proof: $P(E_k|A) = \dfrac{P(E_k \cap A)}{P(A)}$ ← conditional probability

Reversed

$$= \frac{P(A|E_k)\,P(E_k)}{P(A)}$$

$$= \frac{P(A|E_k)\,P(E_k)}{\sum_{i=1}^{n} P(A|E_i)\,P(E_i)}$$ ← Law of total probability

$\boxed{\rightarrow \text{Ex. Part b}}$

Example: Two Coins – One Fair, One Two Headed.

(a) If I randomly select a coin and flip it twice, what is the probability I get heads both times?

Ans. Let E_1 = Fair coin selected
E_2 = Two headed coin selected

$P(E_1) = ?\quad 0.5$
$P(E_2) = ?\quad 0.5$
} If random selection → equal opportunity

↳ Sample space for coin flips? Possible outcomes?
HH, HT, TH, TT

$P(HH \mid E_1) = ?\quad 0.25$ ⇒ Fair coin
$P(HH \mid E_2) = ?\quad 1.0$

$$\Rightarrow P(HH) = \sum_i P(HH \mid E_i)\, P(E_i)$$
$$= P(HH \mid E_1)\, P(E_1) + P(HH \mid E_2)\, P(E_2)$$
$$= 0.25\,(0.5) + 1.0\,(0.5) = 5/8$$

↑ Applying law of total probability

(b) Now, what if I randomly select a coin and flip it twice. If I get heads both times, what is the probability I had the two headed coin?

$P(E_2|HH) = ?$ → Bayes Theorem

$$P(E_2|HH) = \frac{P(HH|E_2)P(E_2)}{\sum P(HH|E_i)P(E_i)} \leftarrow \text{part a}$$

$$= \frac{P(HH|E_2)P(E_2)}{P(HH)}$$

$$= \frac{(1)(1/2)}{5/8} = \underline{\underline{4/5}}$$