CE/CSE 5710: Modeling & Simulation

Applications of Markov Chains[*]

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Success Runs

Consider the probability transition matrix:

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 & \dots \\ p_1 & r_1 & q_1 & 0 & 0 & \dots \\ p_2 & 0 & r_2 & q_2 & 0 & \dots \\ p_3 & 0 & 0 & r_3 & q_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where $p_i > 0$, $q_i > 0$ and $p_i + q_i + r_i = 1$ for $i = 0, 1, 2....$ Consider a situation where a component of a system (a component of a computer, a pipe network segment etc.) has an active life, measured in discrete units, that is given by the random variable T, where $Pr[T = k] = a_k$, for $k = 1, 2...$, where $\sum_{k=1}^{\infty} a_k = 1$. Suppose one starts with a fresh component, and each component is replaced by a new component upon failure (you can also consider a replacement as a service intervention).

Consider the above matrix in the context of such a success run Markov chain, where $r_k = 0$. The age of the component reverts to 0 on failure and given that the age of the component in service is currently k, the failure occurs in the next time period with the following conditional probability:

$$p_k = \frac{a_{k+1}}{a_{k+1} + a_{k+2} + \dots} \tag{1}$$

Therefore q_k is the probability of the component not failing and instead living another day, so $q_k = 1 - p_k$.

Age Replacement Policy

Following from the previous section, let X_n be the age of the component in service at time n. Hence at time of failure X_n is set to 0 by definition. Consider the policy that calls for replacement of the component at age N or when it fails, which ever occurs first. Then X_n has a success run Markov chain with following $N \times N$ probability transition matrix (state space is defined on the set of integers from $0 - (N - 1)$):

$$\mathbf{P} = \begin{bmatrix} p_0 & 1-p_0 & 0 & 0 & 0 & \dots & 0 \\ p_1 & 0 & 1-p_1 & 0 & 0 & \dots & 0 \\ p_2 & 0 & 0 & 1-p_2 & 0 & \dots & 0 \\ p_3 & 0 & 0 & 0 & 1-p_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[*]Material in this chapter is adapted from H. M. Taylor and S. Karlin. An Introduction to Stochastic Modeling. Academic Press, 3rd. edition, 1998., Chapters III and IV.

As state 0 corresponds to a new unit π_0 is the long run probability of replacement during any time unit or the long run replacement per unit time. At $X_n = N - 1$ a planned replacement occurs. Therefore, π_{N-1} is the long run planned replacement per unit time. Therefore rate of failure of service is the difference between π_0 and π_{N-1}. Solving the following system of equations for the above matrix:

$$p_0\pi_0 + p_1\pi_1 + \dots + p_{N-2}\pi_{N-2} + \pi_{N-1} = \pi_0 \quad (2)$$

$$(1 - p_0)\pi_0 = \pi_1 \quad (3)$$

$$(1 - p_1)\pi_1 = \pi_2 \quad (4)$$

$$(1 - p_{N-2})\pi_{N-2} = \pi_{N-1} \quad (5)$$

$$\pi_0 + \pi_1 + \dots + \pi_{N-2} + \pi_{N-1} = 1 \quad (6)$$

When solved this leads to:

$$\pi_0 = \frac{1}{A_1 + A_2 + A_3 \dots + A_n} \quad (7)$$

$$where \qquad 1 - p_k = \frac{A_{k+2}}{A_{k+1}} \quad (8)$$

$$and \qquad \pi_{N-1} = A_N\pi_0 \quad (9)$$

Therefore if the cost of replacement is C and an additional cost of K is incurred when a failure in service occurs, then the long run total cost per unit time is $C\pi_0 + K(\pi_0 - \pi_{N-1})$. Choose the replacement age N to minimize this cost. What is the mean time between replacements?