CE/CSE 5710: Modeling & Simulation

Stochastic Processes

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Stochastic Processes

A Stochastic Process is defined as a collection of random variables $X_t = x$, where t is the parameter running over an index set T. The value $x \in S$ where S is the state space or the range of all possible values that the process can be in, and which the random variable X can take up.

In the discrete case, the index t represents discrete time points where the index set T is an ordered collection of time points, $T = \{0, 1, 2, ...\}$ Hence, $X_t = x$ means that the random variable X had the value x at the time point t, where $t \in T$. Written as $\{X_t, t = 0, 1, ...\}$.

In the continuous case the index t represents a time interval defined on the interval $T = [0, \infty)$. The process is said to be in the state $X = x$ during the interval t, where $t \subset T$. Written as $\{X_t, t > 0\}$.

A stochastic process is not a deterministic system and the values taken by the random variables are expressed probabilistically. They can distinguished by the following:

- State space: Discrete or continuous, finite or infinite
- Index set T: Discrete or continuous, typically $T = [0, \infty)$
- Relationship between successive random variables and the influence of the index parameter.

For our purposes we will consider stochastic processes that have the Markov property.

Markov property

The Markov Property is a property of a stochastic process that implies that if the current state in the process is known exactly, then the probability of its future behavior is not altered by any additional knowledge concerning its past behavior. In other words, the future is independent of the past, given the present, or that the future and the past are conditionally independent.

Stochastic processes with the Markov property can be characterized as:

- Markov Chains: Discrete state space and discrete index parameter (typically time)
- Continuous Time Markov Chains: Discrete state space and continuous index parameter (typically time)
- Discrete Markov Process: Discrete state space and continuous index parameter (typically time)
- Markov Process: Continuous state space and continuous index parameter (typically time)

A *discrete time Markov Chain* is a Markov process whose state space is finite and discrete and the time index is discrete and infinite. The Markov property can be expressed as:

$$Pr\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, ...X_{n_1} = i_{n_1}, X_n = i\} \tag{1}$$

$$= Pr\{X_{n+1} = j | X_n = i\} \tag{2}$$

for all time points n and states $i_0, ...i, j$. This probability is written as:

$$P_{i,j}^{n,n+1} = Pr\{X_{n+1} = j | X_n = i\} \tag{3}$$

When this probability is the same between the states i and j for any n and $n+1$ - or the probability is independent of time (index variable), then the process is said to be stationary. It is referred to as the one step transition probability and written as:

$$P_{i,j} = Pr\{X_{n+1} = j | X_n = i\} \tag{4}$$

Problem 1

A Markov chain $X_0, X_1, X_2...$ has the transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

Initial Distribution: $p_0 = 0.5, p_1 = 0.5$
Calculate:

1. $P_\gamma \ \{X_2 = 1, X_3 = 1 | X_1 = 0\}$.
2. $P_\gamma \ \{X_1 = 1, X_2 = 1 | X_0 = 0\}$.

Problem 2

Consider the Markov Chain $\{X_n\}$ whose transition probability matrix is:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix}$$

Let T = $\min\{n \geq 0;\ X_n = 0 \text{ or } X_n = 2\}$. T is defined as the time to absorption.
Find $u = \Pr\ \{X_T = 0 | X_0 = 1\}$ and $v = E\{T | X_0 = 1\}$