CE/CSE 5710: Modeling & Simulation

*Applications of Markov Chains**

February 17, 2012

Short Term Cash Management

Context: Maintain cash balance to meet future disbursements. Eliminate idle cash balances by investing in short-term loans, treasury bills etc., but maintain a minimum balance to cover potential cash shortages. **Objective:** Identify the best cash management strategy.

In the absence of intervention the cash level in the account fluctuates randomly as a result of multiple small transactions. Time is divided into successive equal length periods of short duration. X_n = Cash in hand at the end of period n and it is assumed to be a random walk in which:

$$Pr\{X_{n+1} = k \pm 1 | X_n = k\} = \frac{1}{2} \tag{1}$$

The cash management strategy is defined by an (s, S) policy as in the inventory problem $(0 < s < S)$. If the cash levels drop below 0 then sell enough treasury bills to replenish cash level up to s, similarly if it increases past S then invest in treasury bills and bring down cash levels to s.

The following can be derived:

- Let T denote the time period at which the cash level either reaches 0 or S at which points it is reset to s. Therefore, T is the time to transaction. The following formula denotes the mean time to transaction or the mean cycle length:

$$E[T|X_0 = s] = v_s = s(S - s) \tag{2}$$

- Given an arbitrary state k, W_{sk} is the mean number of visits to k up to time T starting at $X_0 = s$

$$W_{sk} = 2[\frac{s}{S}(S - k) - (s - k)^+] \tag{3}$$

- Therefore the total unit periods of cash on hand, W_s, up to time T is given by:

$$W_s = \sum_{k=1}^{S-1} k W_{sk} = \frac{s}{3}[S^2 - s^2] \tag{4}$$

In developing a method to identify the best strategy, we first develop a cost metric that accounts for the cost of each transaction and the cost of not investing cash on hand - i.e., the opportunity cost. The following items are defined:

- T_i is duration of the ith. cycle. As each cycle starts from cash level of s - successive cycles are independent.

*Material in this chapter is adapted from H. M. Taylor and S. Karlin. An Introduction to Stochastic Modeling. Academic Press, 3rd. edition, 1998., Chapters III, Section 6.2.

- K is the cost of each transaction - incurred at the end of each cycle.
- R_i is the total opportunity cost of holding cash on hand during the ith. cycle.
- r denotes the cost per unit time of cash on hand.

Therefore, average cost (C) over n cycles can be represented as:

$$C = \frac{nK + \sum_{i=1}^{n} R_n}{\sum_{i=1}^{n} T_n} \tag{5}$$

For very large values of n or as $n \to \infty$, dividing the numerator and denominator of the above expression by n yields:

$$C = \frac{nK + E[R_i]}{E[T_i]} \tag{6}$$

where the $E[R_i]$, the expected value of R_i is expressed as $E[R_i] = rW_s$ and $E[T_i]$ the expected value of each cycle is $E[T_i] = v_s$. Therefore, the long run average cost is (substituting Eq 2 and Eq 4 in Eq 6):

$$C = \frac{K + \frac{1}{3}rs(S^2 - s^2)}{s(S - s)} \tag{7}$$

Eq **??** can be re-written in terms of $x = \frac{s}{S}$ as follows:

$$C = \frac{K + \frac{1}{3}rS^3x(1 - x^2)}{S^2x(1 - x)} \tag{8}$$

The ideal strategy would be one that minimizes the above cost. Therefore, setting the following conditions to minimize C

$$\frac{dC}{dx} = 0 \tag{9}$$

and,

$$\frac{dC}{dS} = 0 \tag{10}$$

yields optimal values, $x_{opt} = \frac{1}{3}$ and $S_{opt} = 3s_{opt} = 3\sqrt[3]{\frac{3K}{4r}}$